Exercise 1

Use residues to derive the integration formulas in Exercises 1 through 5.

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right) \quad (a > b > 0).$$

Solution

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$(z^2 + a^2)(z^2 + b^2) = 0$$
$$z^2 + a^2 = 0 \quad \text{or} \quad z^2 + b^2 = 0$$
$$z = \pm ia \quad \text{or} \quad z = \pm ib$$

The singular points of interest to us are the ones that lie within the closed contour, $z = ib$ and $z = ia$.

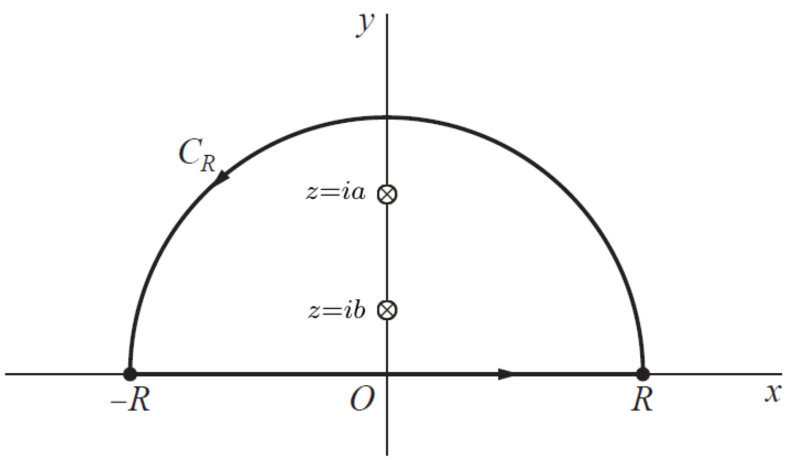

Figure 1: This is Fig. 99 with the singularities at $z = ib$ and $z = ia$ marked.

According to Cauchy's residue theorem, the integral of $e^{iz}/[(z^2 + a^2)(z^2 + b^2)]$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \, dz = 2\pi i \left[\operatorname*{Res}_{z=ib} \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} + \operatorname*{Res}_{z=ia} \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \right]$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \, dz + \int_{C_R} \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \, dz$$
$$= 2\pi i \left[\operatorname*{Res}_{z=ib} \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} + \operatorname*{Res}_{z=ia} \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)} \right]$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L:&\quad z = r, & r = -R \quad &\to \quad r = R \\ C_R:&\quad z = Re^{i\theta}, & \theta = 0 \quad &\to \quad \theta = \pi \end{aligned}$$

As a result,

$$\begin{aligned} \int_{-R}^{R} \frac{e^{ir}}{(r^2+a^2)(r^2+b^2)}\,dr + \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz \\ = 2\pi i \left[\operatorname*{Res}_{z=ib} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} + \operatorname*{Res}_{z=ia} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\right]. \end{aligned}$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{e^{ir}}{(r^2+a^2)(r^2+b^2)}\,dr = 2\pi i \left[\operatorname*{Res}_{z=ib} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} + \operatorname*{Res}_{z=ia} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\right]$$

The denominator can be written as $(z^2+a^2)(z^2+b^2) = (z+ia)(z-ia)(z+ib)(z-ib)$. From this we see that the multiplicities of the $z - ia$ and $z - ib$ factors are both 1. The residues at $z = ib$ and $z = ia$ can then be calculated by

$$\begin{aligned} \operatorname*{Res}_{z=ib} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} &= \phi_1(ib) \\ \operatorname*{Res}_{z=ia} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} &= \phi_2(ia), \end{aligned}$$

where $\phi_1(z)$ and $\phi_2(z)$ are equal to $f(z)$ without the $z - ib$ and $z - ia$ factors, respectively.

$$\begin{aligned} \phi_1(z) &= \frac{e^{iz}}{(z+ia)(z-ia)(z+ib)} \quad \Rightarrow \quad \phi_1(ib) = \frac{e^{i^2b}}{(ib+ia)(ib-ia)(2ib)} = \frac{e^{-b}}{2ib(a^2-b^2)} \\ \phi_2(z) &= \frac{e^{iz}}{(z+ia)(z+ib)(z-ib)} \quad \Rightarrow \quad \phi_2(ia) = \frac{e^{i^2a}}{(2ia)(ia+ib)(ia-ib)} = -\frac{e^{-a}}{2ia(a^2-b^2)} \end{aligned}$$

So then

$$\begin{aligned} \operatorname*{Res}_{z=ib} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} &= \frac{e^{-b}}{2ib(a^2-b^2)} \\ \operatorname*{Res}_{z=ia} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} &= -\frac{e^{-a}}{2ia(a^2-b^2)} \end{aligned}$$

and

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{ir}}{(r^2+a^2)(r^2+b^2)}\,dr &= 2\pi i \left[\frac{e^{-b}}{2ib(a^2-b^2)} - \frac{e^{-a}}{2ia(a^2-b^2)}\right] \\ \int_{-\infty}^{\infty} \frac{\cos r + i \sin r}{(r^2+a^2)(r^2+b^2)}\,dr &= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \\ \int_{-\infty}^{\infty} \frac{\cos r}{(r^2+a^2)(r^2+b^2)}\,dr + i\int_{-\infty}^{\infty} \frac{\sin r}{(r^2+a^2)(r^2+b^2)}\,dr &= \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right). \end{aligned}$$

Therefore, matching the real and imaginary parts of both sides and changing the dummy integration variable to x,

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+a^2)(x^2+b^2)}\,dx = 0.$$

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}
\int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz &= \int_0^{\pi} \frac{e^{iRe^{i\theta}}}{[(Re^{i\theta})^2+a^2][(Re^{i\theta})^2+b^2]}(Rie^{i\theta}\,d\theta) \\
&= \int_0^{\pi} \frac{e^{iR(\cos\theta + i\sin\theta)}}{(R^2e^{i2\theta}+a^2)(R^2e^{i2\theta}+b^2)}(Rie^{i\theta}\,d\theta) \\
&= \int_0^{\pi} \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{(R^2e^{i2\theta}+a^2)(R^2e^{i2\theta}+b^2)}(Rie^{i\theta}\,d\theta)
\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}
\left|\int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| &= \left|\int_0^{\pi} \frac{e^{iR\cos\theta}e^{-R\sin\theta}}{(R^2e^{i2\theta}+a^2)(R^2e^{i2\theta}+b^2)}(Rie^{i\theta}\,d\theta)\right| \\
&\le \int_0^{\pi} \left|\frac{e^{iR\cos\theta}e^{-R\sin\theta}}{(R^2e^{i2\theta}+a^2)(R^2e^{i2\theta}+b^2)}(Rie^{i\theta})\right| d\theta \\
&= \int_0^{\pi} \frac{\left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right|}{\left|R^2e^{i2\theta}+a^2\right|\left|R^2e^{i2\theta}+b^2\right|}\left|Rie^{i\theta}\right| d\theta \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta}}{\left|R^2e^{i2\theta}+a^2\right|\left|R^2e^{i2\theta}+b^2\right|}R\,d\theta \\
&\le \int_0^{\pi} \frac{e^{-R\sin\theta}}{(|R^2e^{i2\theta}|-|a^2|)(|R^2e^{i2\theta}|-|b^2|)}R\,d\theta \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta}}{(R^2-a^2)(R^2-b^2)}R\,d\theta \\
&= \int_0^{\pi} \frac{e^{-R\sin\theta}}{\left(1-\frac{a^2}{R^2}\right)\left(1-\frac{b^2}{R^2}\right)}\frac{d\theta}{R^3}
\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| \le \lim_{R\to\infty}\int_0^{\pi} \frac{e^{-R\sin\theta}}{\left(1-\frac{a^2}{R^2}\right)\left(1-\frac{b^2}{R^2}\right)}\frac{d\theta}{R^3}$$

Because the limits of integration do not depend on R, the limit may be brought inside the integral.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| \le \int_0^{\pi} \lim_{R\to\infty}\frac{e^{-R\sin\theta}}{\left(1-\frac{a^2}{R^2}\right)\left(1-\frac{b^2}{R^2}\right)}\frac{d\theta}{R^3}$$

Since θ lies between 0 and π, the sine of θ is positive. Thus, the exponent of e tends to $-\infty$, and the integral tends to zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| \le 0$$

The magnitude of a number cannot be negative.

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz\right| = 0$$

The only number that has a magnitude of zero is zero. Therefore,

$$\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = 0.$$